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# More memory under evolutionary learning may lead to chaos

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## Abstract

We show that an increase of memory of past strategy performance in a simple agent-based innovation model, with agents switching between costly innovation and cheap imitation, can be quantitatively stabilising while at the same time qualitatively destabilising. As memory in the fitness measure increases, the amplitude of price fluctuations decreases, but at the same time a bifurcation route to chaos may arise.

*Keywords:* Heterogeneous agent models, Imitation, Innovation, Memory, Stability

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## 1. Introduction

Economic and, more generally, social systems of many interacting decision makers have been fruitfully studied with tools from statistical physics [1, 2, 3]. With this approach, their emergent properties can often be addressed as if dealing with a dynamical system of much lower dimensionality [4].

In agent based economic modelling the introduction of memory in the agents' decision mechanisms is usually thought of as having a stabilising effect on the economic system, in the sense that it reduces price variability over time [5, 6, 7, 8, 9]. Intuitively, memory has a smoothing effect on variables entering the price equation, leading to reduced amplitudes of price fluctuations.

In this paper we show that such a quantitative stabilisation may in fact be accompanied by a qualitative destabilisation, since, although memory may reduce the variance of prices, the dynamics can become more irregular or even chaotic. We demonstrate this using the cobweb demand-supply model with discrete choice dynamics introduced by [10], reducing it to the simplest case of two types of agents with rational expectations, differing only in their cost structure.

This paper is structured as follows. Section 2 presents a simple benchmark economic dynamic model. Section 3 introduces memory in the model and studies its effects. Section 4 concludes.

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## 2. The model

Consider an industry with a large number of firms producing the same good in a perfectly competitive market, as in [10]. However, instead of considering agents who can choose from a number of different price predictors, consider the case where all agents have rational expectations but differ in the cost structure of the technology they use to produce the good. In particular, there are two available strategies: *innovation*, which requires an investment but also leads to a reduction in production cost, and *imitation*, which amounts to keeping the currently available technology. The relative fraction of *innovators* in period  $t$ ,  $t \in \mathbb{N}$ , is denoted by  $n_t \in [0, 1]$ , while the fraction of *imitators* is  $1 - n_t$ . The quantity  $S^h(p_t)$ ,  $h \in \{\text{inn}, \text{im}\}$ , where ‘inn’ represents innovation and ‘im’ imitation, supplied in period  $t$  by a firm choosing strategy  $h$  is a function of price and depends itself on the cost structure of strategy  $h$ . The total supply is a fraction-weighted convex combination of the supply by the innovators and the imitators. In each period  $t$  the market clears according to a Walrasian temporary market equilibrium in which demand equals supply, that is,

$$D(p_t) = n_t S_t^{\text{inn}}(p_t) + (1 - n_t) S_t^{\text{im}}(p_t). \quad (1)$$

The profits  $\pi_t^h$  of a firm of type  $h$  in period  $t$  are  $\pi_t^h = p_t q_t^h - c^h(q_t^h)$ , with  $q_t^h \equiv S_t^h(p_t)$ . As in [11], a quadratic cost function is assumed; the cost of producing quantity  $q$  for a firm adopting strategy  $h$  is  $c^h(q) = \frac{q^2}{2s^h} + C^h$ , where  $C^h$  represents the fixed costs of the strategy. This keeps the model as simple as possible, since maximisation of profits with respect to quantity  $q$  gives rise to a linear supply function

$$S_t^{\text{inn}}(p_t) = s^{\text{inn}} p_t, \quad S_t^{\text{im}}(p_t) = s^{\text{im}} p_t. \quad (2)$$

The parameters  $s^{\text{inn}}$  and  $s^{\text{im}}$  are inversely proportional to the marginal production cost, and can be thought of as the productivity of innovators and imitators, respectively. An innovator invests  $C^{\text{inn}} = C > 0$  and increases productivity by a factor  $e^{bC}$ , where  $b > 0$  represents the benefits of the innovation investment:  $s^{\text{inn}} = s e^{bC}$ . An imitator is left with  $s^{\text{im}} = s$ .

Following [10], we assume evolutionary selection or reinforcement learning of strategies. More precisely, we assume that firms switch between two strategies, innovation versus imitation, based on a measure  $U_t$  of past performance, and use the discrete choice model according to which the fractions are determined by the logistic equation

$$n_t = \frac{e^{\beta U_{t-1}^{\text{inn}}}}{e^{\beta U_{t-1}^{\text{inn}}} + e^{\beta U_{t-1}^{\text{im}}}} = \frac{1}{1 + e^{-\beta \Delta U_{t-1}}}, \quad (3)$$

where  $\Delta U_t \equiv U_t^{\text{inn}} - U_t^{\text{im}}$ . The larger (smaller) the difference in past performance  $\Delta U_{t-1}$  between the two strategies, the more (less) firms will decide to innovate. The intensity of choice  $\beta$  measures how sensitive firms are to performance differences. For  $\beta = 0$  agents split equally among the different types. On the other hand,  $\beta = \infty$  represents the limit where all agents choose the strategy with the best past performance.

Without memory, the performance measure  $U_t^h$  coincides with the profit  $\pi_t^h$  for type  $h$  realised in the last period, i.e.  $U_t^h = \pi_t^h$ . For the quadratic cost function these profits are

$$\pi_t^{\text{inn}} = \frac{1}{2}s^{\text{inn}}p_t^2 - C, \quad \text{and} \quad \pi_t^{\text{im}} = \frac{1}{2}s^{\text{im}}p_t^2. \quad (4)$$

The difference in performance between the two strategies is then  $\Delta U_t = \pi_t^{\text{inn}} - \pi_t^{\text{im}} \equiv \Delta \pi_t = \frac{1}{2}s(e^{bC} - 1)p_t^2 - C$ . In particular  $\Delta \pi = 0$  for  $p = \bar{p} \equiv \sqrt{\frac{2C}{s(e^{bC} - 1)}}$ .

For analytical tractability, we consider a linear demand function  $D(p_t) = a - dp_t$  ( $d > 0$ ). The market equilibrium equation (1) becomes

$$a - dp_t = n_t s^{\text{inn}} p_t + (1 - n_t) s^{\text{im}} p_t. \quad (5)$$

Solving for  $p_t$  we obtain

$$p_t = \frac{a}{d + se^{bC}n_t + s(1 - n_t)} \equiv f(n_t). \quad (6)$$

The function  $f$  is decreasing in the fraction  $n_t$ , since  $e^{bC} > 1$ , which means that *a larger number of innovators is associated to a lower market price*. When everybody innovates the price reaches its minimum value  $p_{\text{inn}}^* = a/(d + se^{bC})$ . On the other hand, the maximum market price is  $p_{\text{im}}^* = a/(d + s)$ , when there are only imitators, as illustrated in Fig. 1. The more innovators, the steeper the aggregate supply curve and the lower the price. This

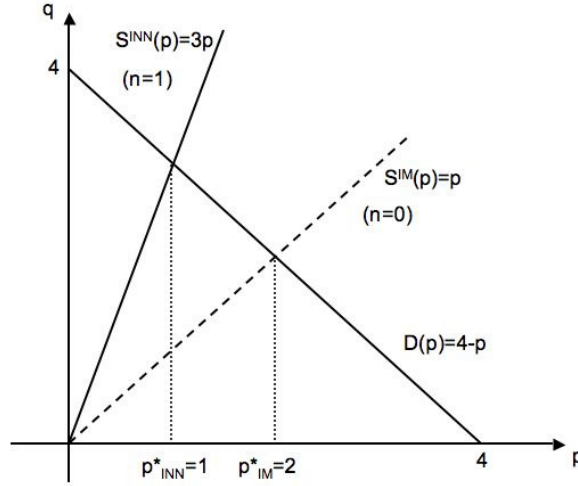


Figure 1: Demand and supply curves with  $D(p) = 4 - p$ ,  $s^{\text{inn}} = 3$  and  $s^{\text{im}} = 1$ . This gives  $S(p) = n^{\text{inn}}3p + n^{\text{im}}p$ . With only innovators the price is  $p_{\text{inn}}^* = 1$ , while with only imitators it is  $p_{\text{im}}^* = 2$ .

mechanism results in a minority game dynamics [12, 13] as we infer from agents' profits (4): a lower price hurts innovators more than imitators, so that innovating is more attractive when less agents are innovators. Equivalently, imitating is better in a market dominated by innovators. Using (3) and (4) we have

$$n_t = \left(1 + e^{-\beta \left[\frac{1}{2}s(e^{bC} - 1)p_{t-1}^2 - C\right]}\right)^{-1}. \quad (7)$$

If we substitute (6) into (7), we obtain a one-dimensional system:

$$n_t = \left( 1 + e^{-\beta \left\{ \frac{s(e^{bC} - 1)a^2}{2[d + se^{bC}n_{t-1} + s(1-n_{t-1})]^2 - C} \right\}} \right)^{-1} \equiv g(n_{t-1}). \quad (8)$$

An equilibrium is a fixed point  $n^*$  of the function  $g$ . Since  $g$  is decreasing, there exists only one equilibrium, which can be either locally stable or locally unstable. In the latter case we have periodic dynamics with period 2. The equilibrium is locally stable whenever  $-1 < g'(n^*) < 0$ , and typically it becomes locally unstable for large values of  $\beta$  [14].

The intuition behind the period 2 dynamics is that innovation drives down the price, and at some point the profits from innovation become too low due to the fixed costs  $C$ , so that imitation becomes preferable. A net flow towards imitative behaviour starts, and the price goes up. The increasing price boosts the innovators' profits more than the imitators', because of their larger productivity. When the innovator's profits become larger than the imitator's, most agents switch back to innovation, and the story is repeated. This cyclical behaviour reflects a "minority game" dynamics: the strategy adopted by the minority turns out to be more profitable. Hence, there is a negative feedback due to the endogenous strategy adaptation mechanism.

### 3. Memory

Next we extend the model of the previous subsection by assuming that agents also take into account profits  $\pi_s$ ,  $s \leq t$ , from the further past. The performance measure is [10]

$$U_t^h = wU_{t-1}^h + (1 - w)\pi_t^h. \quad (9)$$

with  $w \in [0, 1)$ . The parameter  $w$  determines the degree of memory. For larger values of  $w$  the past profits are weighted relatively high compared to the last observed profit  $\pi_{t-1}$ . For  $w = 0$  we are back at the basic model, and only profits from the previous period are taken into account.

We will show that the introduction of memory has a quantitatively stabilising effect and a qualitatively destabilising effect in this model. Note that the system with memory is still one-dimensional. The difference in performance between innovators and imitators evolves according to

$$\Delta U_t = w\Delta U_{t-1} + (1 - w)\Delta \pi_t. \quad (10)$$

Using Eqs. (3) and (6) one can express the difference of realised profits  $\Delta \pi_t = \frac{1}{2}s(e^{bC} - 1)p_t^2 - C$  as a function of  $\Delta U_{t-1}$ :

$$\begin{aligned} \Delta \pi_t &= \frac{s(e^{bC} - 1)a^2}{2 \left[ d + s \left( 1 + \frac{e^{bC} - 1}{1 + e^{-\beta \Delta U_{t-1}}} \right) \right]^2} - C \\ &\equiv h(\Delta U_{t-1}), \end{aligned} \quad (11)$$

where  $h(\Delta U_{t-1})$  is decreasing in  $\Delta U_{t-1}$ . If we substitute Eq. (11) into Eq. (10) we obtain a one-dimensional map

$$\begin{aligned} \Delta U_t &= w\Delta U_{t-1} + (1 - w)h(\Delta U_{t-1}) \\ &\equiv H_w(\Delta U_{t-1}), \end{aligned} \quad (12)$$

describing the system in terms of the state variable  $\Delta U_t$ . Notice that  $H_w$  is the convex

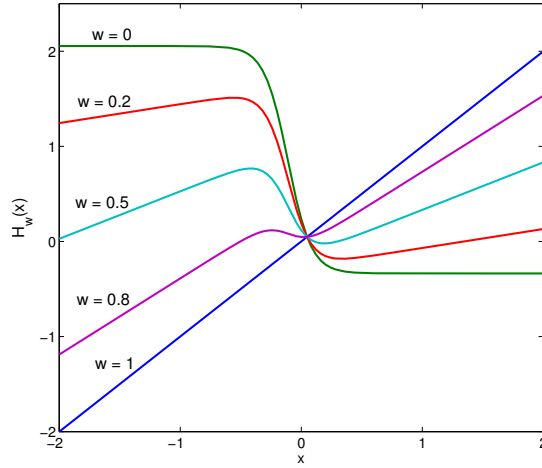


Figure 2: Graph of the map of differential performance  $\Delta U_t = H_w(\Delta U_{t-1})$  for different values of the memory weight  $w$  ( $\beta = 10$ ,  $C = 1$ ,  $b = 1$ ,  $s = 2$ ,  $a = 4$   $d = 1$ ).

combination of an increasing and a decreasing function, which results in a non-monotonic map for intermediate values of  $w$ . This is illustrated in Fig. 2, which shows a number of maps for different values of  $w$ . [15] showed similar non-monotonic maps for the cobweb model with homogeneous adaptive learning. Fig. 3 shows bifurcation diagrams for memory weights  $w$  between 0 and 1 for three different values of the intensity of choice  $\beta$ . In all cases it can

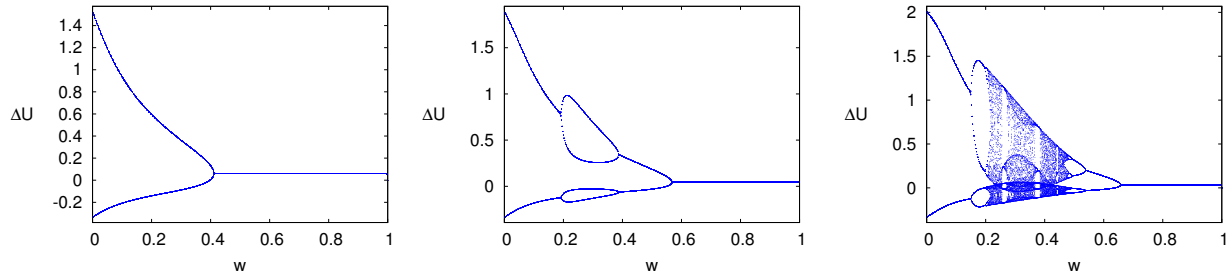


Figure 3: Bifurcation diagrams of the memory weight  $w$  for  $\Delta U$ . Top:  $\beta = 7$ . Centre:  $\beta = 11$ . Bottom:  $\beta = 15$ . Other parameters are  $C = 1$ ,  $b = 1$ ,  $s = 2$ ,  $a = 4$  and  $d = 1$ .

be observed that as the degree of memory  $w$  increases the amplitude of the stable two-cycle decreases. For values of  $w$  close to 1, with much weight given to prices in the distant past, stable dynamics arise. When the intensity of choice to switch strategies is high (bottom panel), the decreasing amplitude of the fluctuations is accompanied by a period-doubling route to chaos, leading to chaotic fluctuations for intermediate values of  $w$ .

The core mechanism leading to the chaotic behaviour in this cobweb discrete choice model is that the map obtained for the system with memory is a convex combination of an increasing linear function and a decreasing non-linear function.

## 4. Conclusions

Using a simple discrete choice model we have shown that memory can have a qualitatively destabilising effect, beside the commonly acknowledged quantitative stabilisation. The value of the intensity of choice is critical in this respect, with higher strategy switching intensity leading to irregular behaviour and a bifurcation route to chaos.

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